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A CLASS OF COMPLETE SOLUTIONS FOR BENDING OF PERFECTLY-PLASTIC BEAMS*

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Abstract—An approach is developed for determining complete solutions for simply supported rectangular beams composed of a rigid, perfectly-plastic material under the action of distributed surface load. Detailed solutions are given for loading over a central strip in plane strain, and the solutions exhibit a sharp transition from failure in bending to failure by local plastic flow at the Prandtl value of the applied pressure. The influence of axial force is discussed and some complete solutions are developed for bending of simply supported beams in plane stress. Comparisons are made between the exact values of the collapse pressure and the values given by beam theory.

1. INTRODUCTION

THE small number of exact or complete solutions to problems within the framework of the theory of plasticity reflects the difficulties involved. Even in the case of plane strain of rigid, perfectly-plastic bodies at the state of collapse when the solution can in principle be obtained from the well developed theory of the slip-line field (see [1, 2] for example), exact solutions are few—especially for problems which involve rigid regions in conjunction with plastically deforming regions. Consequently many investigations have used the theorems of limit analysis in order to obtain bounds on the collapse load of a rigid, perfectly-plastic structure and other investigations employ approximate structural theories such as beam, plate and shell theories. Further, assessment of the validity of the structural theory for beams [3–5] and for beams, plates and shells [6] has been made through the use of limit analysis rather than by means of exact solutions. Although structural theories and the theorems of limit analysis can give close values for collapse loads, the details of the stress field and the deformation mode in the plastically deforming region can only be found from a complete two- or three-dimensional solution.

In the following work we develop a general approach for determining complete solutions for a rigid, perfectly-plastic beam of rectangular section which is simply supported at its ends and which is loaded by both a transverse load and by an axial load. In contrast to engineering beam theory we must specify the end conditions for the beam exactly, and in general we have taken a uniform distribution of shear stress on the ends although it is apparent that complete solutions can be derived for other supporting distributions of shear. Finally we restrict ourselves to plane strain conditions, approximated by a wide

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beam, and to plane stress conditions, approximated by a narrow beam; in the latter case we restrict ourselves to the von Mises yield condition.

In Section 2 we discuss stress fields and associated velocity fields for simply supported beams in plane strain when the loading applied to the upper surface of the beam is non-localized in nature. Detailed solutions are presented for the case of loading by uniform pressure over a central strip of width $2c$, where $c/h > 1/2$, h being the beam thickness. In this case the velocity field involves plastic flow in a conventional yield hinge with straight slip-lines inclined at 45° to the horizontal—the stress fields in the upper and lower parts of the hinge being uniform. The exact value of the collapse pressure is found to be somewhat higher than the value given by beam theory. The solutions for non-localized loading over a central strip apply as long as the value of the collapse pressure is less than $0{\cdot}404\,\sigma_0$, σ_0 being the tensile yield stress. When the pressure is uniform over the whole of the beam this restriction merely requires that the thickness to length ratio $h/2l$ be less than 0·415. When $h/2l = 0{\cdot}415$ the collapse pressure is 17 % greater than the beam theory value.

Section 3 again considers in detail the case when the loading is uniform over a central strip of width $2c$ but now the loading is localized, $c/h < 1/2$. In the velocity fields of these solutions the usual form of yield hinge does not appear. However plastic stress fields and associated velocity fields can be constructed from the theory of the slip-line field and illustrations of two such slip-line fields are given. It is clear from these slip-line fields that as c/h decreases the transition from failure due to bending to failure due to local plastic flow occurs suddenly at the Prandtl value $(1+\pi/2)\sigma_0$ of the applied pressure, where σ_0 is again the tensile yield stress. This agrees with previous results of the authors, who investigated this transition by approximate means with the aid of the limit analysis theorems [6], and obtained close bounds on the collapse pressure. For c/h less than $0{\cdot}1\,h/l$ approximately, the collapse pressure is constant at the Prandtl value. In contrast beam theory predicts a virtually constant total load, which implies very large pressures for small c/h.

In Section 4 we briefly discuss the development of complete solutions when axial force is present in the problems of Sections 2 and 3. Finally in Section 5 we consider bending of simply supported beams in plane stress for a material which obeys the von Mises yield condition. The case of uniform pressure applied over the entire upper surface of the beam is treated in detail and it is clear that bending by a strip of pressure can be handled in much the same fashion as non-localized strip loading in plane strain. The exact value of the collapse pressure is again found to be somewhat higher than the value given by beam theory, as found previously [5] by the use of limit analysis.

2. BENDING OF A BEAM IN PLANE STRAIN

We consider a simply supported beam in plane strain of depth h and length $2l$ which is at collapse under a distributed pressure $q(x)$ applied over the upper surface of the beam. When the pressure loading is not highly localized failure will occur in bending with a yield hinge at the section of maximum moment. (We assume that the maximum moment is attained at only one section.) The general features of the solution are indicated in Fig. 1. A fully plastic state in which the axial stress σ_x is compressive extends from the upper surface of the beam. The slip-line field in this region is determined by the known pressure distribution $q(x)$ on the upper surface together with the equilibrium equations and the yield condition for plane strain

$$(\sigma_x-\sigma_y)^2+4\tau_{xy}^2 = \sigma_0^2, \tag{2.1}$$

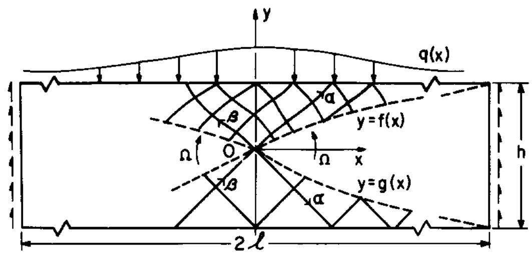

FIG. 1. Simply supported beam under distributed pressure.

the yield strength in simple tension being σ_0 for a Tresca material and $\sqrt{3}\,\sigma_0/2$ for a von Mises material. When $q(x)$ is non-uniform the slip-lines are not straight. For the lower portion of the beam the axial stress is tensile and the traction-free lower surface determines the fully plastic state extending from the lower surface to be simply $\sigma_x = \sigma_0, \sigma_y = \tau_{xy} = 0$; here the slip-lines are straight lines inclined at 45° to the bottom surface. The plastic regions meet at a point O which we take to be the origin of the (x, y) coordinate system. For symmetric loading the section $x = 0$ through O is the section of maximum moment but otherwise the section of maximum moment is close to $x = 0$. The stress distribution across $x = 0$ must be such that there is no net axial force on $x = 0$ and for the portion of the beam to the right of $x = 0$ the moment about O on $x = 0$ must balance the moment about O of the applied pressure $q(x)$ and the shear on the end. These conditions will determine the distance of O from the upper surface and also the beam thickness h in terms of l (or alternatively the collapse value of a loading parameter).

The stress fields in the upper and lower portions of the beam can be extended throughout the beam by introducing lines of stress discontinuity $y = f(x)$ and $y = g(x)$ from O to the corners of the beam as shown in Fig. 1. In the transition region between $y = f(x)$ and $y = g(x)$ we take the equilibrium stress field

$$\sigma_x = 0, \quad \tau_{xy} = \tau(x), \quad \sigma_y = -y\tau'(x)+\sigma(x) \tag{2.2}$$

which satisfies the condition of zero normal stress and uniform shear stress on the ends of the beam. The functions $\tau(x)$ and $\sigma(x)$ as well as the positions of the lines of stress discontinuity are determined by the equilibrium requirements across $y = f(x)$ and $y = g(x)$. Stress fields of this type have been used in [6].

An admissible velocity field can be associated with the stress field described above. The plastically deforming region, the yield hinge, consists of the two triangular regions in the upper and lower portions bounded by the slip-lines from O to the upper and lower surfaces, the α- and β-lines of Fig. 1. The parts of the beam to the left and right of the hinge remain rigid and perform rigid body rotations of amount Ω about O. The velocity field in the hinge is determined from the Geiringer equations for the velocities and the known normal velocities on the hinge boundaries. The solution described here satisfies the requirements for a complete solution provided that the stress field (2.2) for the region lying between $y = f(x)$ and $y = g(x)$ does not violate the yield condition (2.1). For conventional beams the field (2.2) will not violate the yield condition for a wide range of pressure distributions $q(x)$ and complete solutions will be obtained. Finally we remark that the results of [7] can be used to show that the stress field in the plastically deforming region, the yield hinge, will be unique.

In order to exhibit a solution in detail we now consider the case when the pressure $q(x)$ is uniform over a central strip of width $2c$ and zero elsewhere on the upper surface. In this section we suppose that $c/h > 1/2$, approximately, and the case of localized loading, $c/h < 1/2$, will be treated in the next section. We assume that no normal force acts on the ends $x = \pm l$ of the beam, and for definiteness we assume that the shear on the ends is uniformly distributed, although it will be apparent that other shear distributions can be treated.

Referring to Fig. 2, the y-axis is the axis of symmetry and we divide the right-hand half of the beam into regions I–V separated by stress discontinuities, the curves $y = f(x)$, $y = g(x)$ and the straight lines AB and AD. The line AD is inclined at an angle $(\pi/4 - \delta)$ to the x-axis and angle DAB is a right angle. In region I directly under the loaded portion of the beam we take initially a state of hydrostatic pressure of amount q and this satisfies the normal pressure requirement on the upper surface. The pressure q on AD is supplied by a shaft of uniform compressive stress of amount q directed along AB in region IV. Initially region V is stress free. It is easily seen that equilibrium across AB and AD is satisfied. The final stress fields for the region above $y = f(x)$ are obtained by superimposing a uniform axial compression $\sigma_x = -\sigma_0$ on the stress field of regions I, IV and V. Since the angle δ is at our disposal we can select δ so that for a given q the field in DAB is at yield. Then, from (2.1), δ and q are related by

$$q = 2\sin 2\delta \tag{2.3}$$

and the constant state field in region IV is then given by

$$\begin{aligned} \sigma_x &= -X = -1-\tfrac{1}{2}q+\tfrac{1}{4}q^2 \\ \sigma_y &= -Y = -\tfrac{1}{2}q-\tfrac{1}{4}q^2 \\ \tau_{xy} &= T = \tfrac{1}{2}q(1-\tfrac{1}{4}q^2)^{1/2} \end{aligned} \tag{2.4}$$

where we have taken $\sigma_0 = 1$. Finally, as described earlier, in region II we take $\sigma_x = 1$, $\tau_{xy} = \sigma_y = 0$ and in region III the stress field (2.2).

The value of the collapse pressure q and the distance b of O from the upper surface of the beam are determined from the overall equilibrium of the part $x \geq 0$ of the beam. The requirement of no net axial force on $x = 0$ gives $b = h/(2+q)$ while moment equilibrium about O determines the value of the collapse pressure to be (with $\sigma_0 = 1$)

$$q = \frac{h^2}{2c(2l-c)} + \left\{1+\left[\frac{h^2}{2c(2l-c)}\right]^2\right\}^{1/2} - 1. \tag{2.5}$$

Since h/l is small for typical beams, we can expand the square root to give

$$q = \frac{h^2}{2c(2l-c)}\left\{1+\frac{h^2}{4c(2l-c)}+O\left(\frac{h^6}{c^3[2l-c]^3}\right)\right\}. \tag{2.6}$$

The distances of the points D and B of Fig. 2 from the axis of symmetry are denoted by a_1 and a_2 respectively. For the stress fields taken in regions I, II and III, equilibrium conditions across the stress discontinuity $y = f(x)$ require that

$$\begin{aligned} (q+1)f'(x) - \tau(x) &= 0 \\ \tau(x)f'(x) + \tau'(x)f(x) - \sigma(x) - q &= 0 \end{aligned} \tag{2.7}$$

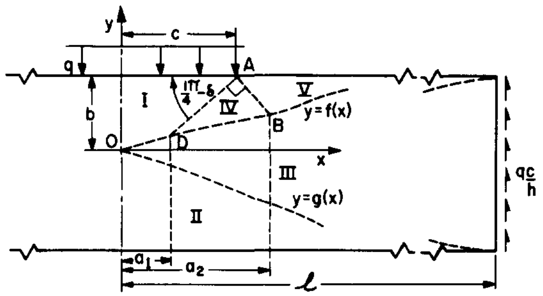

FIG. 2. Simply supported beam loaded by a central strip of uniform pressure, $c/h \geq c_1/h \simeq 1/2$.

and similarly for $y = g(x)$ we need

$$\begin{aligned} g'(x)+\tau(x) &= 0 \\ \tau(x)g'(x)+g(x)\tau'(x)-\sigma(x) &= 0. \end{aligned} \tag{2.8}$$

The system of differential equations (2.7) and (2.8) is valid for $0 < x < a_1$ and has the initial conditions $f(0) = g(0) = 0$. Addition of the first equations and subtraction of the second equations of (2.7) and (2.8) lead to

$$g(x) = -(1+q)f(x)$$
$$\tau(x)[f(x)-g(x)] = qx$$

on integration and use of the initial conditions. Substituting for $\tau(x)$ from (2.7) then gives

$$(2+q)(1+q)f'(x)f(x) = qx$$

and a final integration gives

$$f(x) = \left\{\frac{q}{(1+q)(2+q)}\right\}^{1/2} x. \tag{2.9}$$

From (2.7), (2.8) and (2.9) we find that region III is a constant state region for $0 < x < a_1$ with $\tau_{xy} = -g'(x)$, $\sigma_x = 0$, $\sigma_y = -[g'(x)]^2$. The yield condition (2.1) with $\sigma_0 = 1$ is therefore not violated as long as $|g'| \leq 0{\cdot}486$. This restriction then gives $q = 0{\cdot}404$ as the maximum value of q for which the yield condition (2.1) is not violated in region III, $0 < x < a_1$.

For $a_1 < x < a_2$, (2.2) and (2.4) together with the equilibrium requirements across $y = f(x)$ give

$$\begin{aligned} Xf'(x)+T-\tau(x) &= 0 \\ [\tau(x)-T]f'(x)-Y+f(x)\tau'(x)-\sigma(x) &= 0 \end{aligned} \tag{2.10}$$

while for equilibrium across $y = g(x)$ equations (2.8) again apply. The functions $f(x)$ and $g(x)$ are continuous at $x = a_1$ and the continuity of $\tau(x)$ at $x = a_1$ then gives the remaining initial conditions

$$g'(a_1) = -\tau(a_1), \qquad f'(a_1+0) = [\tau(a_1)-T]/X$$

from the first equations in (2.8) and (2.10). The shear at $x = a_1$, $\tau(a_1)$, can be computed from the previous results for $x < a_1$ and is found to be

$$\tau(a_1) = \{q(1+q)/(2+q)\}^{1/2}.$$

From (2.8) and (2.10) and the conditions at $x = a_1$ it is found that $f(x)$ and $g(x)$ are determined in the interval $a_1 < x < a_2$ by

$$\begin{aligned} Xf(x)+g(x) &= -(x-a_1)T - Yf(a_1) \\ Xf^2(x)+g^2(x) &= (x-a_1)^2 Y + 2(x-a_1)A + B \end{aligned} \tag{2.11}$$

where

$$A = Xf(a_1)f'(a_1+0)+g(a_1)g'(a_1), \qquad B = Xf^2(a_1)+g^2(a_1).$$

In a similar fashion the functions $f(x)$ and $g(x)$ can be determined for $x > a_2$. Without giving the details we find that

$$\begin{aligned} f(x) &= \{\tfrac{1}{4}h^2 - qc(l-x)\}^{1/2} - \tfrac{1}{2}\frac{qh}{2+q} \\ g(x) &= -f(x) - \frac{qh}{2+q}. \end{aligned} \tag{2.12}$$

The shear at the end $x = l$ has the correct value $\tau(l) = qc/h$ and the curves $y = f(x)$ and $y = g(x)$ pass through the corners of the beam. Finally, the values of a_1 and a_2 can be determined from the preceding results but the calculations are not given here.

We remark that the line of stress discontinuity AB of Fig. 2 can be replaced by a fan region (of small angle) centered at A such that the stress field changes continuously from that of region IV to the uniform compressive field of region V. The value of the collapse pressure given by (2.5) is unaffected by this change, of course, although the positions of the lines of stress discontinuity, $y = f(x)$ and $y = g(x)$, are altered slightly. Although the stress discontinuity AB is used here in order to simplify the calculations of this section, the fan centered at A must be taken for c/h close to 1/2 to ensure that the stress fields change smoothly into the stress fields of the next section for $c/h < 1/2$ (approximately).

The stress field discussed here is statically admissible provided that $q \leq 0{\cdot}404$, which guarantees that region III of Fig. 2 is below yield (the shear is a maximum in the portion $0 < x < a_1$ of region III), and provided that $c \geq c_1$ where c_1 is the value of c when D coincides with O. If $\delta = \delta_1$ and $q = q_1$ when $c = c_1$ then c_1 is determined from

$$c_1 = \frac{h}{(2+q_1)}\cot(\tfrac{1}{4}\pi - \delta_1). \tag{2.13}$$

If we expand the cotangent about $\delta_1 = 0$ and employ (2.3) we find that $c_1/h = \frac{1}{2}+O(q^2)$. The restriction $q \leq 0{\cdot}404$ will be satisfied when h/l lies in the range of practical values (and $c \geq c_1$). When the loading is over the whole of the beam ($c = l$), for example, the restriction $q \leq 0{\cdot}404$ merely requires $h/l \leq 0{\cdot}83$.

As discussed previously we can associate an admissible velocity field with the statically admissible stress fields of Fig. 2. The hinge region in which plastic deformation takes place is bounded by the 45° lines $y = \pm x$ through O. The rest of the beam remains rigid and performs rigid rotations of amount Ω about O. Within the hinge, the velocity components (u, v) are given by

$$u(x, y) = \begin{cases} -\Omega x, & y > 0 \\ \Omega x, & y < 0, \end{cases} \qquad v(x, y) = \begin{cases} \Omega y, & y > 0 \\ -\Omega y, & y < 0. \end{cases} \tag{2.14}$$

This velocity field satisfies the incompressibility requirement, is continuous across the rigid–plastic boundaries $y = \pm x$, and the directions of maximum shear strain-rate are along 45° lines in the x- and y-plane which coincide with the directions of maximum shear stress for the stress fields of regions I and II. Since, in addition, the shear stress and shear strain-rate have the same sign, the flow rule for the yield condition (2.1) is satisfied and our solution is therefore complete for $q \leq 0{\cdot}404$ and $c \geq c_1$. The value (2.5) is therefore the exact value of the collapse pressure.

We remark that the beam theory with the moment yield condition $|M| \leq M_0 = h^2/4$ $(\sigma_0 = 1)$ gives the collapse pressure $q = h^2/[2c(2l-c)]$. Equation (2.6) shows that the exact value of the collapse pressure determined from the complete solution is always greater than the beam theory value for the range $c \geq c_1$ although they approach each other as h/l goes to zero. Figure 3 illustrates the difference in the beam theory and exact values of collapse pressure as a function of c/h for a beam whose thickness to span ratio $h/2l$ is one-tenth. The collapse pressure given by (2.5) is valid for $c/h \geq c_1/h = 0{\cdot}499$ when $h/2l = 1/10$.

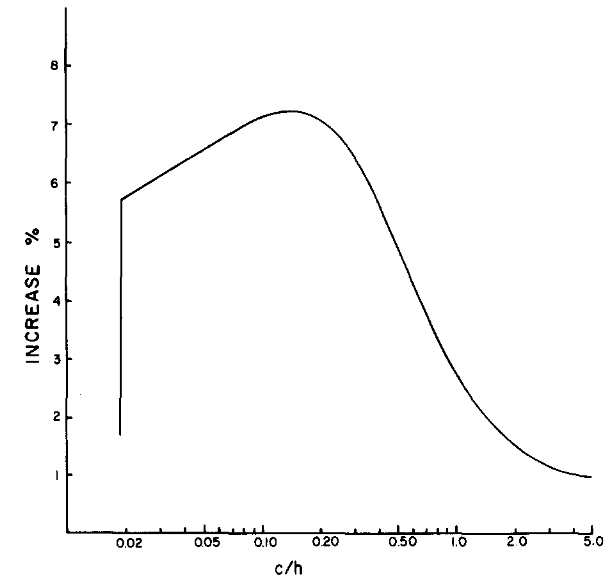

FIG. 3. Percentage increase of the exact value of the collapse pressure over the beam theory value for loading over a central strip ($h/2l = 1/10$).

3. LOCALIZED LOADING OF A BEAM IN PLANE STRAIN

In this section we develop complete solutions for localized pressure loading, that is $c/h < 1/2$ approximately, where $2c$ is again the width of the central strip over which the uniform pressure q is applied. We shall see that three types of solution are sufficient to describe the beam behavior when $c/h < 1/2$, and the complete solutions developed illustrate the sharp transition from failure in bending to local plastic flow at the Prandtl value $(1+\pi/2)\sigma_0$ of the applied pressure.

When c/h is not too small, the various stress regions are as indicated in Fig. 4. The regions are separated by stress discontinuities along the curves $y = f(x)$ and $y = g(x)$ and the straight lines AC and CF. The line CA is inclined at an angle $(\pi/4-\gamma)$ to the horizontal while ACF is a right angle. The y-axis is again taken at the axis of symmetry while the position of the x-axis will be fixed by the requirement of no net axial force on $x = 0$. In region BAC directly under the applied load we take the stress field $\sigma_x = -1-q$, $\sigma_y = -q$, $\tau_{xy} = 0$ (with $\sigma_0 = 1$) while in region COF we take $\sigma_x = -1-r$, $\sigma_y = -r$, $\tau_{xy} = 0$. The slip-lines for these fully plastic fields are inclined at 45° to the x- and y-axes as shown in Fig. 4. Region ACFE is a fully plastic constant state region while DAE is a centered fan

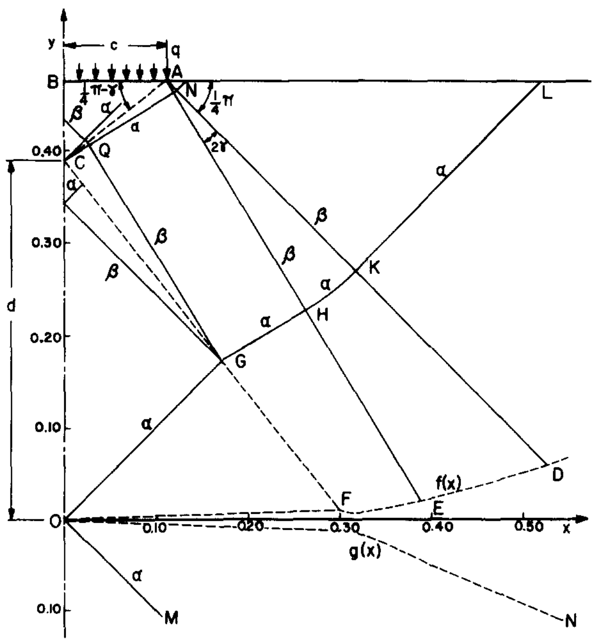

FIG. 4. Development of slip-line field for a beam under localized loading with $2\gamma = 14°$ and $h/2l = 1/10$.

region with fan angle 2γ and DA inclined at 135° to the x-axis. In the region to the right of AD for $y \geq f(x)$ we have $\sigma_x = -1$, $\sigma_y = \tau_{xy} = 0$ while for $y \leq g(x)$ we take the axial tensile field $\sigma_x = 1$, $\sigma_y = \tau_{xy} = 0$. In the central region bounded by $y = f(x)$ and $y = g(x)$ and to the left of DN we take the equilibrium stress field

$$\sigma_x = -s(y), \quad \tau_{xy} = \tau(x), \quad \sigma_y = -y\tau'(x)+\sigma(x) \tag{3.1}$$

where now σ_x is taken to be non-zero in case high values of shear $\tau(x)$ occur in this region. In the central region to the right of DN σ_x is gradually reduced to zero in order to satisfy the condition of zero normal stress on $x = l$. Equilibrium across AC and CF is satisfied if

$$q = 2\gamma + \sin 2\gamma$$
$$r = 2\gamma - \sin 2\gamma \tag{3.2}$$

and the stress field in ACFE is then given by

$$\sigma_x = -X = -\tfrac{1}{2} - 2\gamma - \tfrac{1}{2}\cos 4\gamma$$
$$\sigma_y = -Y = -\tfrac{1}{2} - 2\gamma + \tfrac{1}{2}\cos 4\gamma \tag{3.3}$$
$$\tau_{xy} = T = \tfrac{1}{2}\sin 4\gamma.$$

Overall equilibrium of the right half $x \geq 0$ of the beam must be satisfied. From the requirement of no net axial force we obtain

$$(1+q)c\tan(\tfrac{1}{4}\pi - \gamma) + (1+r)d = h - c\tan(\tfrac{1}{4}\pi - \gamma) - d \tag{3.4}$$

while moment equilibrium about O gives

$$(1+q)c\tan(\tfrac{1}{4}\pi - \gamma)[d + \tfrac{1}{2}c\tan(\tfrac{1}{4}\pi - \gamma)]$$
$$+\tfrac{1}{2}(1+r)d^2 + \tfrac{1}{2}[h - c\tan(\tfrac{1}{4}\pi - \gamma) - d]^2 + \tfrac{1}{2}qc^2 = qcl. \tag{3.5}$$

From (3.4) and (3.5) we get, on eliminating d,

$$c^2[(2+r)q - (2+q)(q-r)\tan^2(\tfrac{1}{4}\pi - \gamma)]$$
$$-2c[(2+r)ql - (q-r)h\tan(\tfrac{1}{4}\pi - \gamma)] + h^2(1+r) = 0. \tag{3.6}$$

Equations (3.2) and (3.6) are sufficient to determine the dependence of the collapse pressure q on c since a choice of γ determines q and r in (3.2) and then (3.6) can be solved to determine c/h for a given thickness to span ratio $h/2l$. For small values of γ we can simplify (3.6). To order γ^3 we see from (3.2) that $r = 0$ and $q = 4\gamma$, and substitution of these values into (3.6) leads to the following expression for q,

$$q = \frac{h^2}{4lc} + \frac{h^3}{8l^2c} + O\left(\frac{h^3}{l^3}\right) \tag{3.7}$$

which expresses the dependence of q on h/l as h/l goes to zero for fixed values of h/c. To order h^3/l^3 the value q given by (3.7) agrees with the value (2.6) when $c/h = 1/2$.

The stress fields developed for the fully plastic regions of the beam can be extended throughout the beam in much the same fashion as in Section 2. In fact if we take $s(y) = 0$ in (3.1) then (2.7)–(2.9) apply with q replaced by r in the analysis and equations (2.10) and (2.11) again apply with X, Y and T given by (3.3) and where a_1 and a_2 are now the distances of the points F and E from the y-axis. In this fashion we determined the position of the curves $y = f(x)$ and $y = g(x)$ up to $x = a_2$ for the case when the fan angle 2γ equaled 14° and the thickness to span ratio $h/2l$ was 1/10 so that from (3.2) and (3.6) we have $q = 0{\cdot}486$, $r = 0{\cdot}002$ and $c/h = 0{\cdot}112$. The curves $y = f(x)$ and $y = g(x)$ are shown to scale in Fig. 4. The positions of the curves were also determined for $x \geq a_2$ through the fan region by numerical integration of the differential equations expressing equilibrium over $y = f(x)$ and $y = g(x)$—the determination of the positions of the curves of stress discontinuity to the right of section DN is straightforward (see (2.12)). It was found that the yield condition

(2.1) was not violated in the central region for the selection $s(y) = 0$ in (3.1) with $2\gamma = 14°$ and $h/2l = 1/10$. The stress fields developed here are therefore statically admissible.

We can associate a velocity field directly with the stress field discussed for the localized loading in question. Referring again to Fig. 4 we assume a deformation symmetric about $x = 0$ in which plastic deformation is confined to the region between $x = 0$ and the α-lines through O, OGHKL in the upper part of the beam and OM in the lower part of the beam. The rigid portion of the beam rotates counterclockwise about O with angular velocity Ω. In the lower part of the plastic hinge the velocity components (u, v) referred to the x- and y-axes are given for $y < 0$ by (2.14). In the upper part of the hinge the determination of the velocity field is straightforward using $u = 0$ on OB as well as the known normal velocity on OGHKL. In fact, the field in the upper part of the hinge consists of the rigid motion $u = -\Omega y$, $v = \Omega x$ with a superposed deformation mode determined by $u = \Omega y$ on $x = 0$ and $u_\beta = 0$ on OGHKL, where we now take u_α and u_β to be the velocity components directed along the α- and β-lines. Since the rate of extension along a stress discontinuity is zero, for the deformation mode the velocity component directed along GC is zero while the velocity component directed along CA is $\Omega d \sec(\pi/4-\gamma)$. With these boundary conditions we can determine the velocity field in region OGC by the method used by Lee for determining velocity fields for an acute angled wedge with the normal velocity prescribed on one face [8]. The velocities on CG then determine the field in the triangular region bounded by CG, the α-line through C, and the β-line through G, region GCQ of Fig. 4. This field can then be extended to the region bounded by GQ and the α-lines QN and GHKL since u_α is known on QG while $u_\beta = 0$ on GHKL. Finally, the velocity field can be extended into regions CNA and ABC.

In this fashion we can demonstrate the existence of an admissible velocity field which satisfies the incompressibility condition as well as the requirement that the maximum shear strain-rate and the maximum shear stress agree in direction and sign. Our solution is therefore a complete solution and the value of the collapse pressure given by (3.2) and (3.6) is the exact value provided the stress field in the central region does not violate the yield condition. From the work of Anderson and Shield [6] in which very close lower and upper bounds were developed on the collapse pressure for highly localized loading, there seems little doubt that $s(y)$ in (3.1) can be selected so that this requirement is met. The form of complete solution described here applies until c is so small that points H and G of Fig. 4 coincide which occurs when $c = c_2$, where

$$c_2 = \tfrac{1}{2}\sqrt{2}d\frac{\sin\gamma\cos(\pi/4-\gamma)}{\cos^2\gamma}. \tag{3.8}$$

As before, the exact value of the collapse pressure given by the complete solution and approximated by (3.7) is always greater than the value furnished by beam theory, although the values approach each other as h/l goes to zero for fixed values of c/h. The variation with c/h of the difference between the beam theory value and the exact value for the range $c_1/h \geq c/h \geq c_2/h$ is included in Fig. 3 for a beam with $h/2l = 1/10$; in this case $c_2/h = 0{\cdot}061$ while $c_1/h = 0{\cdot}499$ from Section 2.

Complete solutions can also be developed for more concentrated loading, $c \leq c_2$, although the associated slip-line fields for the plastically deforming regions can no longer be pieced together from the simple fields used previously and we must resort to numerical or graphical techniques. Figure 5 illustrates the development of the slip-line field under

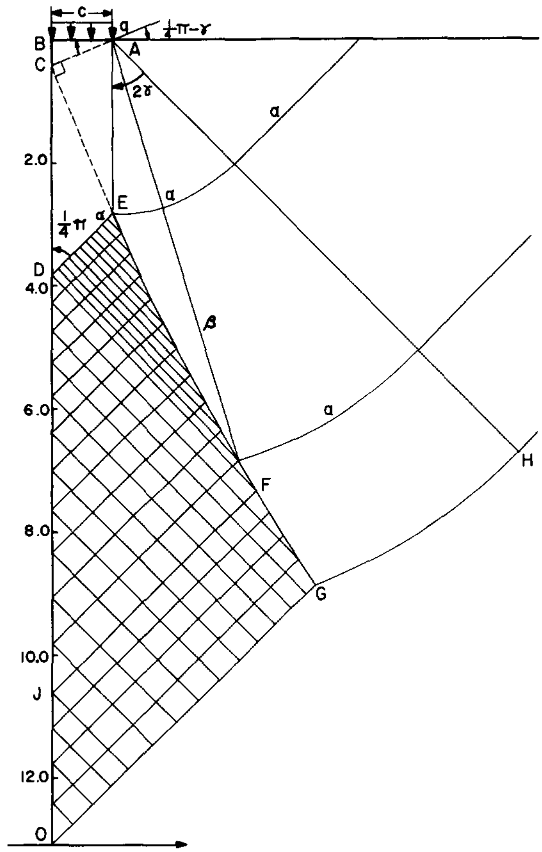

FIG. 5. Development of slip-line field for a beam under localized loading with $2\gamma = 45°$ and $h/2l = 1/10$.

the loaded portion of the beam. The lines AC and CE are again stress discontinuities with AC inclined at an angle $\pi/4-\gamma$ to the horizontal while angle ACE is a right angle. In region BAC we take the stress field $\sigma_x = -1-q$, $\sigma_y = -q$, $\tau_{xy} = 0$ while in region CED we take $\sigma_x = -1-r$, $\sigma_y = -r$, $\tau_{xy} = 0$. Region ACE is a fully plastic constant state region and AE and AH define a fan region with fan angle 2γ and AH inclined at 135° to the horizontal. To the right of AH we again take the compressive field $\sigma_x = -1$, $\sigma_y = \tau_{xy} = 0$. Equilibrium is satisfied across the lines AC and CE when r and q are given by (3.2) and the field in region ACE is again given by (3.3).

The slip-line field for region DEFGO as well as the position of the stress discontinuity EFG are determined by the α-line DE, the condition that the slip-lines meet the axis OD at 45° and the equilibrium jump conditions across EFG. Denoting by θ the inclination of an α-line to the x-axis and setting $p = -(\sigma_x+\sigma_y)/2$, equilibrium requires, in the continuous portion of the field,

$$\begin{aligned} p+\theta &= \text{const.} \quad \text{on an } \alpha\text{-line} \\ p-\theta &= \text{const.} \quad \text{on a } \beta\text{-line}, \end{aligned} \tag{3.9}$$

remembering that we have taken $\sigma_0 = 1$. At the discontinuity EFG, the α-lines must be equally inclined to EFG and also

$$p_1 - p_2 = \sin(\theta_2 - \theta_1) \tag{3.10}$$

where subscripts designate values on either side of EFG (see [1, 2], for example). From the known values $p = 1/2 + r$, $\theta = \pi/4$ on DE, the known stress field in the fan above EFG and the condition $\theta = \pi/4$ on OD, the field can be determined numerically from (3.9), (3.10) in the regions DEF, DFJ, FGOJ in turn. Since DE is straight, the α-lines in DEF are straight lines. Below O the tensile field $\sigma_x = 1$, $\sigma_y = \tau_{xy} = 0$ applies and the slip-line OG must be such that the right half of the beam is in overall equilibrium. In practice, a trial origin O is chosen and the trial beam thickness h is then fixed by the condition of zero axial force across $x = 0$. For a given thickness to span ratio the total moment on the right half of the beam can be found. Adjustment of the location of O is then made to reduce this moment to zero. Figure 5 shows numerical results for a beam with $h/2l = 1/10$. The angle of the fan in Fig. 5 is $2\gamma = 45°$ and it was found that $q = 1{\cdot}49$ and $c/h = 0{\cdot}035$. Again it is anticipated in view of the results of [6] that the stress fields in the upper and lower portions of the beam can be continued throughout the beam without violating the yield condition.

For the velocity field, plastic deformation is confined to the hinge with lower part a right-angled triangle as before and upper part bounded by the α-line OGH and its straight continuation and the reflection of this line in the y-axis. The velocity field in the lower part of the hinge is given by (2.14) while in the upper part the velocity field is determined from the condition $u = 0$ on $x = 0$ and the known normal velocity on OGH (extended), with the subsidiary requirements that AC and CEFG are extensionless.

The bending type solutions described here apply up to the value $2\gamma = \pi/2$ when the discontinuities AC and CE disappear and the well-known Prandtl field is obtained in the vicinity of the loaded region. Any of the velocity fields associated with the Prandtl–Hill solution for the indentation of a half-space by a flat-ended punch then apply (see [1, 2], for example) and $q = 1 + \pi/2$ is an upper bound on the collapse pressure for all smaller values of c/h. The Prandtl value can be shown to be a lower bound also by extending the Prandtl field in the manner used by Shield [9] in constructing lower bounds for the punch problem, with suitable modifications to satisfy the boundary conditions on the lower surface and ends of the beam. Thus, complete solutions can be developed for all values of c/h in the full range $0 < c/h \leq l/h$.

The percentage difference between the beam theory value of the collapse pressure and the value given by the exact theory for $h/2l = 1/10$ is included in Fig. 3 for $c/h \leq c_2/h$, values of 30°, 45° and 60° for 2γ being used in the computation. The sudden drop in the curve is due to the sharp transition from beam behavior to local flow at the Prandtl value of the applied pressure, the beam theory predicting a collapse pressure increasing without bound as c/h tends to zero.

4. THE INFLUENCE OF AXIAL FORCE

The results of the previous two sections can be extended to take into account a distribution of axial force over the ends $x = \pm l$ of the beam, as shown in Fig. 6. Here the beam is loaded uniformly over a central strip of width $2c$, and the load is supported by a uniform distribution of shear on the ends, but other forms of shear distribution and other

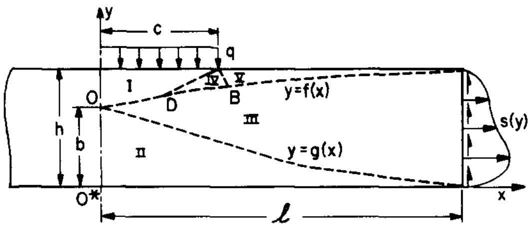

FIG. 6. Simply supported beam loaded by a central strip of uniform pressure with axial force.

pressure loadings can be considered. Following the development of Section 2 we divide the beam of Fig. 6 into regions I–V as before separated by the lines of stress discontinuity shown as broken lines. The stress fields in regions I, IV and V above $y = f(x)$ and the field in region II below $y = g(x)$ are the same as those described in Section 2 for the beam of Fig. 2. In the transition region, region III, we take the field (3.1) where $s(y)$ is now the known distribution of normal stress on $x = \pm l$.

We denote by N the resultant normal force on an end and by y^* the distance of its line of action from the lower surface, which is taken to be the x-axis. Thus we have

$$N = \int_0^h s(y)\,\mathrm{d}y, \qquad Ny^* = \int_0^h ys(y)\,\mathrm{d}y.$$

From overall equilibrium of the portion $x \geq 0$ of the beam we have

$$\begin{aligned} b-(h-b)(1+q) &= N \\ (1+q)(h^2-b^2)-b^2+2Ny^*+qc^2 &= 2qcl \end{aligned} \tag{4.1}$$

and these equations determine the collapse value of the pressure q and the position of O (see Fig. 6). The determination of the stress discontinuities $y = f(x)$ and $y = g(x)$ can be carried out as in Section 2 and again, for conventional beams, the stress field in region III will not violate the yield condition for a wide range of axial force distributions $s(y)$.

A velocity field can be associated with the stress fields described here in which the beam rotates rigidly about O with plastic deformation confined to a yield hinge centered at O and bounded by 45° lines through O. This velocity field provides complete solutions with the exact value of the collapse pressure given by (4.1) up to the value of c such that points O and D of Fig. 6 coincide. For more concentrated loads we can modify the solutions of Section 3 to accommodate the normal force on the ends and thus complete solutions can be developed for all values of c in the full range $0 < c/h \leq l/h$. The solutions will, of course, again exhibit a sharp transition from failure in bending to local flow at the Prandtl value of the collapse pressure.

5. BENDING OF SIMPLY SUPPORTED BEAMS IN PLANE STRESS

The method of solution of Sections 2 and 4 can be adapted with slight modification to bending of simply supported beams in plane stress but where the material of the beam now obeys the von Mises yield condition. Since the construction of stress and velocity fields is very similar to the previous work we shall only consider in detail the problem of bending of simply supported beams under uniform surface load with uniform shear and zero normal stress on the ends $x = \pm l$ of the beam. This problem has been treated previously and bounds on the exact value of the collapse pressure were obtained by use of limit analysis [5]. Here we shall give a complete solution.

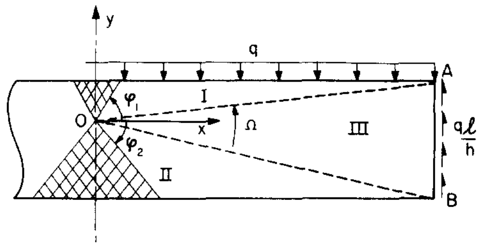

FIG. 7. Simply supported beam in plane stress under uniform surface load.

Referring to Fig. 7, we divide the beam into regions separated by straight lines from O to the corners of the beam, OA and OB, as shown. In regions I, II and III we take the constant stress fields

$$\begin{aligned} &\text{Region I}: \quad \sigma_x = -\lambda - q, \quad \sigma_y = -q, \quad \tau_{xy} = 0 \\ &\text{Region II}: \quad \sigma_x = 1, \quad \sigma_y = \tau_{xy} = 0 \\ &\text{Region III}: \sigma_x = 0, \quad \sigma_y = \sigma, \quad \tau_{xy} = \tau \end{aligned} \tag{5.1}$$

where we have again taken $\sigma_0 = 1$, σ_0 being the yield stress in simple tension. These equilibrium stress fields satisfy the boundary conditions of the problem. The constant λ in (5.1) is chosen so that the von Mises yield condition (with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, $\sigma_0 = 1$) is satisfied in region I, that is

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 = 1 \tag{5.2}$$

and we find that

$$\lambda = -\tfrac{1}{2}q + [1 - 3q^2/4]^{1/2}. \tag{5.3}$$

The values of τ, σ, q and the distance of O from the upper surface of the beam are determined from the equilibrium requirements across OA and OB and we find that O is fixed at a distance $h/(1+\lambda+q)$ below the upper surface, while $\tau = ql/h$, $\sigma = -q^2l^2/h^2$ and q is given by

$$\tfrac{1}{2}q^2 + \left[1 - \tfrac{1}{2}\left(\frac{h}{l}\right)^2\right]q = [1 - 3q^2/4]^{1/2}\left[\left(\frac{h}{l}\right)^2 - q\right]. \tag{5.4}$$

Expanding the square root in (5.4) and then expanding q in a power series in h^2/l^2 gives

$$q = \tfrac{1}{2}\frac{h^2}{l^2}\left\{1 + \tfrac{1}{8}\left(\frac{h^2}{l^2}\right) + O\left(\frac{h^4}{l^4}\right)\right\} \tag{5.5}$$

as the collapse value of q. Using the values of τ and σ together with (5.4) then gives $q = 0{\cdot}56$ as the maximum value of q for which the stress field (5.1) does not violate the von Mises yield condition in region III. For $q = 0{\cdot}56$, $h/2l$ has the value $0{\cdot}51$.

A velocity field can be associated directly with the statically admissible stress field (5.1). The velocity field involves a yield hinge centered at point O of Fig. 7 in which plastic deformation takes place while the rest of the beam remains rigid and performs rigid body rotations of amount Ω about O. With x- and y-axes taken through O as shown, the hinge is bounded for $y \geq 0$ by intersecting lines inclined at an angle φ_1 to the x-axis and for $y \leq 0$ by intersecting lines inclined at an angle φ_2 to the y-axis, where the angles φ_1 and φ_2 are defined by

$$\tan\varphi_1 = \left[\frac{2\lambda+q}{\lambda-q}\right]^{1/2}, \qquad \tan\varphi_2 = \sqrt{2}.$$

The angles φ_1 and φ_2 also define the inclination to the horizontal of the characteristics of the velocity equations in the two parts of the hinge. The velocity components (u, v) within the hinge are given by

$$u(x,y) = \begin{cases} -\left[\dfrac{2\lambda+q}{\lambda-q}\right]^{1/2}\Omega x, & y > 0 \\ \sqrt{2}\Omega x, & y < 0, \end{cases} \qquad v(x,y) = \begin{cases} \left[\dfrac{\lambda-q}{2\lambda+q}\right]^{1/2}\Omega y, & y > 0 \\ -\frac{1}{2}\sqrt{2}\Omega y, & y < 0. \end{cases} \tag{5.6}$$

The velocity field (5.6) is continuous across the rigid–plastic boundary and is related to the stress fields in regions I and II through the flow rule. Thus for $h/2l < 0{\cdot}51$ our solution is complete and the value of the collapse pressure determined by (5.4) is the exact value.

As in the case of plane strain, complete solutions can be derived when the applied load is uniformly distributed over a central width $2c$ and also for the case when axial force is present. However, for values of $p = -\frac{1}{2}(\sigma_x+\sigma_y)$ such that $p^2 > \sigma_0^2$ the equations describing the plane stress problem with the von Mises yield condition change character from hyperbolic type to elliptic type [10]. Thus for small values of c/h the method of approach of this section does not provide complete solutions.

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Résumé—Une méthode d'approche est développée pour établir des solutions complètes pour des poutres rectangulaires à support simple composées d'un matériau parfaitement plastique–rigide sous l'effet d'une charge superficielle distribuée. Des solutions détaillées sont données pour une charge appliquée à une bande centrale dans une déformation en plan, et les solutions font ressortir une transition nette entre la rupture à la flexion et la rupture de fluage pour la valeur Prandtl de la pression appliquée. L'influence de la force axiale est discutée et quelques solutions complètes sont développées pour la flexion de poutres à support simple dans une déformation en plan. Des comparaisons sont faites portant sur les valeurs exactes de la pression de rupture et les valeurs données par la théorie des poutres.

Zusammenfassung—Eine Methode wird entwickelt zur vollständigen Lösung einfach gestützter rechtwinkliger Träger aus idealplastischem Material unter der Einwirkung verteilter Oberflächenbelastungen. Detailierte Lösungen werden gegeben für Belastungen über ein Mittelband in Planbelastung. Die Lösungen haben einen scharfen Übergang von Verformung durch Biegung zu Verformung durch lokales Kriechen bei dem Prandtl'-schen Wert des Druckes. Der Einfluss der Axialkraft wird behandelt, und vollständige Lösungen werden entwickelt für die Biegung einfach gestützter Träger in Planbelastung. Vergleiche zwischen den genauer Werten des Zerstörungsdruckes und der Werte der Trägertheorie werden gemacht.

Абстракт—Рассуждается метод определения полных решений для свободно опертых прямоугольных балок, построенных из твердого идеально-пластического материала и находящихся под влиянием распределенной поверхностной нагрузки. Приводится детальные решения для случая нагрузки центральной полосы, которая находится в плоском деформированном состоянии. Решения показуют внезапный переход от разрушения при изгибе к разрушению при локальном пластическом течении. Это пластическое течение определяется значением Прандтля для приложенной нагрузки. Исследуется влияние осевых сил. Приводятсн некоторые полные решения для случая изгиба свободно опертых балок в плоском напряженном состоянии. Сравниваютсн точные значения при разрушении давлением, со значениями данными из теории балок.